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Paper - CC 2 mathematical Physics
Unit - 9 Topic - Singularities of Analytic funˢⁿ

Singularities of an Analytic funⁿ

A singularity (or a singular point) of an analytic funⁿ $f(z)$ is a point where $f(z)$ ceases to be analytic. We can also say that $f(z)$ is singular or has a singularity at that point. The funⁿ $f(z)$ is said to be singular at infinity if $f\left(\frac{1}{z}\right)$ is singular at $z = 0$. In a given z-plane an analytic funⁿ may have a number of singularities.

Isolated Singularity:- The point z_0 is defined as an isolated singularity of the funⁿ $f(z)$ if $f(z)$ is not analytic in the neighbourhood of $z = z_0$

Types of Singularities:- Analytic funⁿs may have two types of singularities if we restrict ourselves only to single valued functions:

1. Poles or non-essential singularities
2. Essential singularities.

The distinction between these two types of singularities may be explained as follows:

Let $f(z)$ has an isolated

singularity at $z = z_0$, then it can be represented by its Laurent series about z_0,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} \quad \text{—} \; (1)$$

which is valid throughout some neighbourhood of $z = z_0$ (except at $z = z_0$ itself). The last series in (1) is called the principal part of $f(z)$ near $z = z_0$.

Now there are three possibilities:

(i) The principal part of $f(z)$ consists of only finite number of terms. Then equ^n (1) reduces to the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_m}{(z-z_0)^m}$$

where $b_m \neq 0$ and $b_n = 0$ for all $n > m$.

In this case, where the principal part consists of only a finite number of terms, the singularity of the funⁿ $f(z)$ at $z = z_0$ is called a pole and m is called the order of the pole and the funⁿ $f(z)$ is said to have a pole of m^{th} order at the point z_0. Poles of first order (i.e. $m=1$) are called simple poles.

2. The principal part of $f(z)$ consists of infinite number of terms. Then Laurent expansion of $f(z)$ about z_0 is of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} = \sum_{n=-\infty}^{+\infty} A_n (z-z_0)^n$$

(2)

In this case the singularity of the funⁿ $f(z)$ at $z = z_0$ is called the isolated essential singularity.

Any singularity of a single valued analytic funⁿ other than a pole is called an essential singularity. Poles are, by definition isolated singularities. All ~~singularity~~. Singularities which are not isolated are thus essential singularities. An essential singularity may be isolated or not.

3. The principal part contains no term i.e. all the coefficients b_n are zero. Then the Laurent expansion of $f(z)$ about z_0 is of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

obviously the fun^n $f(z)$ is not analytic at $z=z_0$, but can be made analytic there by assigning some value to $f(z)$ at $z=z_0$. Thus the fun^n $f(z)$ has a singularity at $z=z_0$ which can be made to disappear by defining the fun^n suitably. In this case the singularity of the fun^n $f(z)$ at $z=z_0$ is called the removable singularity. Such singularities are not of interest, because they can be removed.

Entire fun^n :- A fun^n which is analytic everywhere in the finite plane is called an entire fun^n. If such a fun^n is also analytic at infinity, it is bounded for all z and from Liouville's theorem it follows that it must be a constant. Hence any entire fun^n which

which is not a constant must singular at ingulr. For example polynomials of at least the first degree e^z, $\sin z$ and $\cos z$ are entire funⁿs and they are singular at infinity.

Meromorphic funⁿ: – An analytic funⁿ whose only singularities in the finite plane are poles is said to be a meromorphic funⁿ.